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U. S. AIR FORCE

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RESEARCH MEMORANDUM

(Uncl) STATISTICAL THEORY OF NAVIGATION EMPLOYING INDEPENDENT INERTIAL AND VELOCITY MEASUREMENTS

P. Swerling and E. Reich

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SUMMARY

This report discusses the problem of optimum determination of position by a navigating device employing independent inertial and velocity measurements. The measurements are assumed to be subject to random errors.

The problem is related to the theory of statistical estimation as well as to filtering theory. Explicit optimum methods of position computation are derived for several special cases. Formulas are derived by means of which the variance of the error in computed position can be determined. The asymptotic variance, for large time of flight, of the error in computed position is discussed.

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LIST OF SYMBOLS

Note: A detailed explanation of certain parts of the notation is given on p. 9.

Symbol	Definition	Page First Used
$x(t)$	vehicle position at time t	2
$v(t)$	vehicle velocity at time t	4
$b(t)$	$x''(t) + \Omega^2 x(t)$	3
x_o	initial position of vehicle	3
x'_o	initial velocity of vehicle	3
$B(t)$	accelerometer dial reading	3
$V(t)$	velocity dial reading	4
X_o	independent estimate of initial position	4
X'_o	independent estimate of initial velocity	4
T	elapsed time since beginning of flight	1
W	defined by Eq. (IV.3)	27
$\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_1, \gamma_2, \gamma_3, \gamma_4$	parameters describing instrument error statistics	6-8
ε	error	2
ξ	defined by Eq. (III.6)	14
η	defined by Eq. (IV.6)	28
$\phi_B(s,t)$	accelerometer error autocorrelation function	5
$\phi_V(s,t)$	velocity dial error autocorrelation function	5
Ω	frequency of Earth's radius pendulum	3
ψ	defined by Eq. (III.3)	13

I. INTRODUCTION

A. BACKGROUND

Considerable study has been devoted in recent years to the problem of navigating airborne vehicles by utilizing acceleration-sensitive devices housed in the vehicle. In principle, if the position and velocity with respect to inertial space at some initial instant $t = 0$ were known exactly, and if the measuring and computing devices in the vehicle functioned without error, it would be possible for a computer in the vehicle to calculate the vehicle's exact position at any instant $t = T > 0$. To decrease the effect of accelerometer errors, systems incorporating a combination of both accelerometers and independent velocity measuring devices (such as doppler radars) have been proposed, and made the subject of a well developed literature.* In this report we will focus our attention on the accuracy achievable with such a combination type of system.

The computed position contains an error due to the errors in the various components of the system--for example, errors in the accelerometer, in the independent measurement of velocity, in gyros or star trackers used to maintain a reference direction, in the altimeter, in initial values of position and velocity, etc. The question naturally arises: In a position computer utilizing independent accelerometer and velocity measurements what is the "best" way to combine the information from these sources to yield a computed position? In order to make a solution of this problem feasible it is necessary to specialize the problem a good deal. Only a few sources of error are

*See the bibliography of Ref. 2.

considered; various other simplifying assumptions are made. The problem thus specialized is still of considerable interest; many of the papers cited make the same, similar, or more restrictive assumptions.

This report will be restricted to the presentation of the theoretical work. Numerical computations based on the theory, and using experimental noise statistics, are being planned, and the resulting data will be presented in a later report. One of the objectives of the numerical work will be to compare the optimum system, which turns out to have time-varying properties, with more conventional non-time-varying systems.

B. GENERAL NATURE OF INPUT DATA TO THE COMPUTER

For expository purposes we will first consider an ideal, error-free system, and then approximate to the actual system by adding random noise.

In the ideal system the vehicle in question is assumed to move in a great circle, at constant altitude above a spherical and non-rotating earth. The task of the computer is to calculate the position $x(t)$ along the great circle.

In what follows capital letters will denote variables which include random error components, and the corresponding lower case letters will denote the same variables without the error components. The error components will be denoted by $\mathcal{E}$ with an appropriate subscript.

We will refer to the variables upon which the course computer operates as "dial readings". Altogether, there are four dial readings (Fig. 1)--accelerometer, velocity, initial position, initial velocity. The first two vary during flight; the latter two are preset before commencement of flight and merely serve as a permanent store of the preset information.

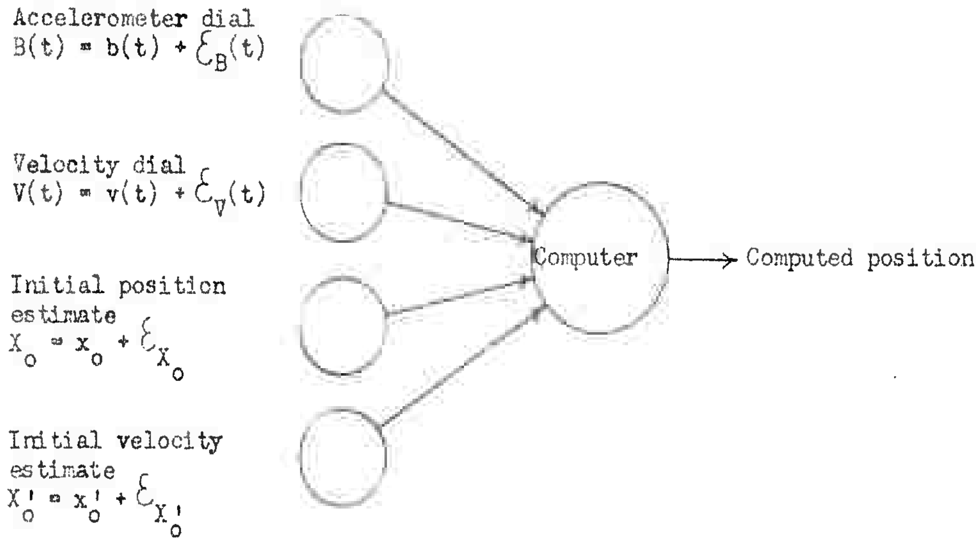

Fig. 1

(a) Accelerometer Dial Reading

As used in this report, the term "accelerometer" is actually somewhat of a misnomer, as the apparatus in question not only serves to indicate the resultant force, but also incorporates a provision for maintaining, by means of gyros and/or star trackers, a fixed direction in inertial space. Theory shows* that, assuming no instrumental errors, the accelerometer dial can be calibrated so that at time t it will read

$$b(t) = x''(t) + \Omega^2 x(t) . \tag{I.1}$$

In line with the remark at the beginning of the paragraph the actual reading $B(t)$ will be assumed to differ from $b(t)$ by an additive noise term $\mathcal{E}_B(t)$:

$$B(t) = b(t) + \mathcal{E}_B(t) = x''(t) + \Omega^2 x(t) + \mathcal{E}_B(t) \tag{I.2}$$

*Cf. e.g. Ref. 1.

(b) Velocity Dial Reading

A true velocity measuring device would indicate

$$v(t) = x'(t) \tag{I.3}$$

The actual reading, however, will be of the form

$$V(t) = v(t) + \varepsilon_V(t) = x'(t) + \varepsilon_V(t) \tag{I.4}$$

(c) Initial Position Estimate

The analysis admits the possibility of an estimate X_o of the initial position x_o being set into the computer before commencement of flight. ("Initial" refers to the instant when the course computer begins to function.) We put

$$X_o = x_o + \varepsilon_{X_o} \tag{I.5}$$

where ε_{X_o} is the error of X_o

(d) Initial Velocity Estimate

In addition to X_o an estimate X_o' of the initial velocity x_o' may be preset into the computer. As above, we put

$$X_o' = x_o' + \varepsilon_{X_o'} \tag{I.6}$$

A short remark apropos X_o and X_o' may serve for proper orientation of

the reader at this point. Every bit of information available to the computer may be expected to aid it in its estimate of position. The computer could estimate x_o and x_o' just on the basis of the accelerometer and velocity dial readings $B(t)$ and $V(t)$ by using the estimate $V(0)$ for x_o' and $\left[B(0) - V'(0)\right]/\Omega^2$ for x_o. X_o and X_o' are additional estimates which are independent of the accelerometer and velocity dial readings. Once the accelerometer and velocity dial readings become available, the computer has essentially two independent estimates of x_o and of x_o' at its disposal. The two estimates of x_o, or of x_o', can be combined (e.g. by a linear weighting) to obtain a new estimate which is better than either of the original ones taken by itself. It is thus clear that (except in degenerate cases) the separate specification of X_o and X_o' is not superfluous, but rather these quantities may be expected to play a role in the optimum position estimation process.

C. <u>ERROR STATISTICS</u>

The accelerometer and velocity dial errors are assumed to be random processes, with autocorrelation functions $\phi_B(s,t)$ and $\phi_V(s,t)$ respectively; and with zero mean over a statistical ensemble of flights. That is, for every instant t,

$$\text{(I.7)} \qquad \overline{\mathcal{E}_B(t)} = 0$$

$$\overline{\mathcal{E}_V(t)} = 0$$

and, for each pair of instants (s,t),

$$\text{(I.8)} \qquad \overline{\mathcal{E}_B(t)\mathcal{E}_B(s)} = \phi_B(s,t)$$

$$\overline{\mathcal{E}_V(t)\mathcal{E}_V(s)} = \phi_V(s,t)$$

The zero mean holds for a statistical ensemble of flights. The possibility exists that on any single flight, the d' l .eadings may be biased. The average value of such a bias will, o: a large number of flights, be zero. The existence of such biases causes a positive constant term to be added to the autocorrelation functions.

The assumption of zero ensemble mean for the errors is clearly no restriction on generality, since if the mean were a known quantity different from zero, it could be subtracted from the dial readings.

The quantities $\mathcal{E}_{X_o}$ and $\mathcal{E}_{X'_o}$ of (I.5) and (I.6) are assumed to be (over an ensemble of flights) random variables with

(I.9) $$\overline{\mathcal{E}_{X_o}} = 0 \; ; \; \overline{\mathcal{E}^2_{X_o}} = \gamma_3 \; ; \; \overline{\mathcal{E}_{X'_o}} = 0 \; ; \; \overline{\mathcal{E}^2_{X'_o}} = \gamma_4$$

The dial errors are all assumed to be statistically independent of each other and of the true path.

D. THE OPTIMIZATION CRITERION

The optimization criterion is as follows: at each instant T , the computer should form an estimate of position $\hat{x}(T)$, based on the dial readings for $0 \leq t \leq T$, such that

a) $\hat{x}(T) - x(T)$, the error in position estimate, is independent of the path. This is equivalent to assuming that no a-priori information--e.g. statistical information--about the set of possible paths is utilized.

b) $\hat{x}(T)$ depends in a linear manner on the dial readings.

c) Expected value (with respect to a statistical ensemble of flights) of $\left[\hat{x}(T) - x(T)\right]^2$ is a minimum for all estimates satisfying (a) and (b).

No further restrictions are made on the possible ways of combining the information contained in the dial readings.

If all statistics entering are Gaussian, the solution to the problem as above formulated is the best of all possible (linear or nonlinear) unbiased* estimates of position.

Conditions (a) and (b) together with the assumed dial error statistics imply that the optimum position estimate will satisfy $\hat{x}(T) = x(T) + \mathcal{E}_{\hat{x}(T)}$, where $\mathcal{E}_{\hat{x}(t)}$ has zero ensemble mean, depends in a linear manner on the dial errors, and does not depend on the path. (Consequently, the system is dynamically exact.)

E. OUTLINE OF THE REMAINDER OF THE REPORT

Section II contains an exposition of the general method of attack on the problem at hand.

It happens that detailed results are easy to obtain in two special cases, viz:

Case 1 $\qquad \phi_B(s,t) = \text{constant} = \gamma_1$

Case 2 $\qquad \phi_V(s,t) = \text{constant} = \gamma_2$

This can be interpreted as follows: either the accelerometer dial (Case 1) or the velocity dial (Case 2) is noiseless except for bias.

Section III is devoted to the treatment of Case 1, Section IV to Case 2. Explicit solutions for the optimum position estimate are obtained only after even further specialization, to wit:

Case 1: Explicit solution obtained for

$$\phi_V(s,t) = \gamma_2 + \beta_2 e^{-\alpha_2 |s-t|} \tag{I.10}$$

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*Over an ensemble of flights.

Case 2: Explicit solution obtained for

$$\phi_B(s,t) = \gamma_1 + \beta_1 e^{-\alpha_1 |s-t|} \tag{I.11}$$

The application of our methods to obtain explicit optimum position estimates for other autocorrelation functions is further discussed in the sections dealing with Cases 1 and 2.

Sections III and IV also contain expressions from which can be obtained the variance of the error in $\hat{x}(T)$ as a function of T.

The results for Cases 1 and 2 are summarized at the end of Sections III and IV, respectively.

A third special case which is treated in Appendix II is

Case 3

$$\begin{cases} \phi_B(s,t) = \beta_1 e^{-\alpha_1 |s-t|} \\ \phi_V(s,t) = \beta_2 e^{-\alpha_2 |s-t|} \end{cases}$$

The solution to this case is relegated to an appendix because, although it is qualitatively interesting, it is so complicated as to be almost useless for purposes of computation. The solution and the method used to obtain it are outlined, but not given in full detail.

Section V shows that the problem of optimum position estimation can be reduced to a linear least squares filtering problem.

Section VI contains a discussion of asymptotic errors.

F. EXPLANATION OF NOTATION

(a) T is the elapsed time since the beginning of the flight.

(b) Dial readings are denoted by B, V, X_o, X'_o. The values these quantities would have, if the dials were errorless, are denoted by b, v, x_o, x'_o.

(c) For $0 \leq t \leq T$, the computer's estimate of x(t) based on all the dial readings up to time T is denoted by $\hat{x}(t;T)$. Thus, $\hat{x}(0;T)$ represents the computer's estimate of initial position based on all dial readings up to time T. The quantity $\hat{x}(T;T)$--i.e. the computer's estimate of position at time T based on the dial readings up to time T--will be denoted by $\hat{x}(T)$.

Similar notation is used for the computer's estimates of other quantities such as velocity: $\widehat{x'}(t;T)$, $\widehat{x'}(0;T)$, etc.*

(d) Errors in all quantities are denoted by $\mathcal{E}$ with an appropriate subscript, as: $\mathcal{E}_B = B - b$; $\mathcal{E}_{\hat{x}(T)} = \hat{x}(T) - x(T)$; etc.

However, errors such as $\mathcal{E}_{B(t)}$, $\mathcal{E}_{\hat{x}(t;T)}$, etc., will usually be denoted by $\mathcal{E}_B(t)$, $\mathcal{E}_{\hat{x}}(t;T)$, etc.

(e) For functions of time F(t), quantities such as $F(t_i)$ will sometimes be denoted by F_i.

(f) The operator $\frac{d^2}{dt^2} + \Omega^2$ is denoted by $\mathcal{L}$: $\mathcal{L}\, x(t) = x''(t) + \Omega^2 x(t)$. The operator $\frac{d}{dt}$ will sometimes be denoted by $\mathcal{D}$: $\mathcal{D}x(t) = x'(t)$.

(g) For a list of the other symbols used, see p. v.

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*In the following pages, reference will often be made to the "best" estimates of x(t), x'(t), etc., based on all the dial readings up to time T, for any t in the range $0 \leq t \leq T$. In such cases, "best" means best in the sense of the optimization criterion of part D (p. 6), with, for position estimation, $\hat{x}(T)$ replaced by $\hat{x}(t;T)$ and x(T) replaced by x(t); for velocity estimation, $\hat{x}(T)$ replaced by $\widehat{x'}(t;T)$ and x(T) replaced by x'(t); etc.

II. METHOD OF SOLUTION

One approach to the problem of finding estimates having minimum error variance is the maximum likelihood method. The role of this method in conventional estimation problems is treated in detail in many standard works.*

The method may be briefly summed up as follows: Consider a vector with random components $\vec{U} = (u_1, \ldots, u_n)$, and suppose that the joint density function of $u_1, \ldots, u_n$ depends on some parameters $\alpha_1, \ldots, \alpha_m$. Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m)$. Let the density function be denoted by $p\left(\vec{U} \middle| \vec{\alpha}\right)$. Then, provided the density function satisfies certain regularity conditions, the unbiased estimates of minimum variance of the parameters $\alpha_1, \ldots, \alpha_m$, based on observed values $\widetilde{U}_1, \ldots, \widetilde{U}_n$, are found by maximizing $p(\widetilde{U}_1, \ldots, \widetilde{U}_n | \alpha_1, \ldots, \alpha_m)$ with respect to $\alpha_1, \ldots, \alpha_m$. The maximizing values of $\alpha_1, \ldots, \alpha_m$ are the required estimates.

In the next two sections, the maximum likelihood method will be applied assuming Gaussian statistics. This will give the estimate of minimum variance among all unbiased estimates. As was pointed out in the introduction, however, the resulting estimate is also the <u>linear</u> estimate of minimum variance, no matter what the noise statistics are, provided the means and correlation functions remain the same.

Another way of expressing this is as follows: for any statistics, the <u>linear</u> estimate of minimum variance can be found by replacing the actual statistics by Gaussian statistics having the same means and correlation functions, and then applying the maximum likelihood method. If the statistics actually are Gaussian, this gives the minimum variance estimate among <u>all</u> unbiased estimates.

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*Cf. e.g. Ref. 3.

The application of the maximum likelihood method to the problem at hand is simpler to explain first for Cases 1 and 2.

In Case 1, for example, $\phi_B(s,t) = \gamma_1$, so that the accelerometer dial is perfect except for bias. This means that the accelerometer error is constant for each flight and equal to the initial accelerometer error.

Now suppose that the velocity dial readings are known only at times $0 = t_o < t_1 < \ldots < t_{n-1} = T$, where t_i are equally spaced over the interval $(0,T)$. The vector $\vec{U} = (B_o, V_o, V_1, \ldots, V_{n-1}, X_o, X_o')$ has a density function which, for Gaussian statistics, can be explicitly written down in terms of $\phi_V(s,t)$, and can be shown (Section III) to depend on just three unknown parameters, which can be taken to be x_o, x_o', and $\mathcal{L}x_o = x_o'' + \Omega^2 x_o$. Also $x(T)$ can be expressed in terms of x_o, x_o', $\mathcal{L}x_o$, and known functions of T.

Therefore, one must maximize $p(\vec{U} \,|\, x_o, x_o', \mathcal{L}x_o)$ with respect to x_o, x_o', and $\mathcal{L}x_o$ (with the observed dial readings as components of $\vec{U}$).

The resulting estimate $\left[\hat{x}(T)\right]_n$ would be the unbiased estimate with minimum error variance, if the velocity dial readings were known only for times t_i. The limit: $\lim_{n\to\infty}\left[\hat{x}(T)\right]_n$ is the required estimate of $x(T)$ based on all information contained in the dial readings for $0 \leqq t \leqq T$.

In Case 2, the velocity dial is perfect except for bias. In this case, the random vector under consideration is $\vec{U} = (B_o, B_1, \ldots, B_{n-1}, V_o, X_o, X_o')$. The density function depends on two unknown parameters, x_o and x_o'; $x(T)$ can be expressed in terms of these and known functions of T. Therefore, one must maximize $p(\vec{U} \,|\, x_o, x_o')$ with respect to x_o and x_o'; obtain $\left[\hat{x}(T)\right]_n$; and take the limit as $n \to \infty$.

In the general case, the problem can in principle be approached as follows: The four dial readings $B(t)$, $V(t)$, X_o, X_o' are shown in Section V

to be equivalent to two dial readings:

$$Q(t) = X_o + \int_o^t V(\tau)\, d\tau$$

$$R(t) = X_o \cos \Omega t + \frac{X_o'}{\Omega} \sin \Omega t + \frac{1}{\Omega}\int_o^t B(\tau) \sin \Omega(t - \tau)\, d\tau$$

in the sense that, knowing $R(t)$ and $Q(t)$ over some interval $0 \leq t \leq T$, one can derive $V(t)$, $B(t)$, X_o, X_o' for that interval; and vice versa.

The readings of the Q and R dials at times $0 = t_o < t_1 < \ldots < t_{n-1} = T$ form a 2n-dimensional random vector, the density function of which depends on the vector $\vec{x} = (x_o, \ldots, x_{n-1})$; here $x_i = x(t_i)$. The autocorrelations and cross correlation of $Q(t)$ and $R(t)$ can be expressed in terms of the B and V autocorrelations and the variances of X_o and X_o'.

The maximum likelihood estimate of $x(T)$ would be obtained by maximizing $p(\vec{U}\,|\,\vec{x})$ with respect to $x_o, \ldots, x_{n-1}$ and then taking $\hat{x}(T) = \lim_{n\to\infty} \left[\hat{x}(T)\right]_n$ where, for each n, $\left[\hat{x}(T)\right]_n = \hat{x}_{n-1}$.

This procedure appears to be of only academic interest in the general case. Section V shows how the problem of optimum position estimation can be reduced to a standard linear least-squares filtering problem. The reason for emphasizing the maximum likelihood approach at this point is simply that this is the approach actually used in obtaining results for Cases 1 and 2.

III. TREATMENT OF CASE 1: $\phi_B(s,t) = \gamma_1$

A. DERIVATION OF OPTIMUM POSITION ESTIMATE

In this case, all sources of error are present except for the non-bias component of accelerometer noise.

Now, since $x''(t) + \Omega^2 x(t) = b(t)$,

$$x(t) = \frac{1}{\Omega}\int_0^t b(\tau) \sin\Omega(t-\tau)\, d\tau + x_o \cos\Omega t + x_o' \frac{\sin\Omega t}{\Omega} \tag{III.1}$$

Since in the present case, $\zeta_B(t) = B(0) - b(0)$,

$$B(t) = b(t) + \zeta_B(t) = b(t) + B_o - \breve{\alpha} x_o \tag{III.2}$$

Let

$$\psi(t) = \frac{1}{\Omega}\int_0^t B(\tau) \sin\Omega(t-\tau)\, d\tau \tag{III.3}$$

Then, putting (III.2) into (III.1),

$$x(t) = \psi(t) + x_o \cos\Omega t + \frac{x_o'}{\Omega} \sin\Omega t + \frac{\breve{\alpha} x_o - B_o}{\Omega^2}(1 - \cos\Omega t) \tag{III.4}$$

$$v(t) = x'(t) = \psi'(t) - x_o \Omega \sin\Omega t + x_o' \cos\Omega t + \frac{\breve{\alpha} x_o - B_o}{\Omega} \sin\Omega t$$

From (III.4) we can say that

$$\hat{x}(t;T) = \psi(t) + \hat{x}(0;T) \cos\Omega t + \frac{\widehat{x'(0;T)}}{\Omega} \sin\Omega t + \frac{\breve{\alpha}\,\widehat{x}(0;T) - B_o}{\Omega^2}(1 - \cos\Omega t) \tag{III.5}$$

(III.5) - cont'd)

$$\hat{v}(t;T) = \widehat{x'}(t;T) = \psi'(t) - \hat{x}(0;T)\,\Omega \sin \Omega t + \widehat{x'}(0;T) \cos \Omega t$$
$$+ \frac{\widehat{\mathcal{L}x}(0;T) - B_o}{\Omega} \sin \Omega t$$

Applying the maximum likelihood method as explained in Section II, one first imagines the velocity dial readings to be given just at times $t_o = 0, t_1, \dots, t_{n-1} = T$. One must then minimize, with respect to $\hat{x}(0;T)$, $\widehat{x'}(0;T)$, and $\widehat{\mathcal{L}x}(0;T)$ the quantity

$$\text{(III.6)} \qquad Q_n = \sum_{i,j=o}^{n-1} \zeta_{ij}\,(V_i - \widehat{x'_i})(V_j - \widehat{x'_j}) + \frac{1}{\gamma_3}\left[X_o - \hat{x}(0;T)\right]^2$$
$$+ \frac{1}{\gamma_4}\left[X'_o - \widehat{x'}(0;T)\right]^2 + \frac{1}{\gamma_1}\left[B_o - \widehat{\mathcal{L}x}(0;T)\right]^2$$

$$\text{where } (\zeta_{ij}) = \left[\phi_V(t_i, t_j)\right]^{-1} \qquad \text{(i.e. the matrix inverse)}$$

Q_n is, apart from a constant independent of $\hat{x}(0;T)$, $\widehat{x'}(0;T)$ and $\widehat{\mathcal{L}x}(0;T)$, equal to $-\ln p\left[U \,\middle|\, \hat{x}(0;T), \widehat{x'}(0;T), \widehat{\mathcal{L}x}(0;T)\right]$.

For convenience let $\hat{x}(0;T) = z_1$; $\widehat{x'}(0;T) = z_2$; $\widehat{\mathcal{L}x}(0;T) - B_o = z_3$. On differentiating Q_n with respect to z_1, z_2, z_3 (after substituting (III.5) into (III.6)) and setting the derivatives equal to zero, three equations are obtained:

$$\text{(i)} \qquad z_1\left(A_{11} + \frac{1}{\gamma_3}\right) - z_2 A_{12} - z_3 \frac{A_{11}}{\Omega^2} = D_1$$

$$\text{(III.7)} \qquad \text{(ii)} \qquad -z_1 A_{12} + z_2\left(A_{22} + \frac{1}{\gamma_4}\right) + z_3 \frac{A_{12}}{\Omega^2} = D_2$$

$$\text{(iii)} \qquad -z_1 A_{11} + z_2 A_{12} + z_3\left(\frac{A_{11}}{\Omega^2} + \frac{\Omega^2}{\gamma_1}\right) = -D_1 + \frac{1}{\gamma_3} X_o$$

where

$$\text{(III.8)}\quad \begin{aligned} &\text{(i)} \quad A_{11} = \Omega^2 \sum_{i,j=o}^{n-1} \xi_{ij} \sin \Omega t_i \sin \Omega t_j \\ &\text{(ii)} \quad A_{12} = \Omega \sum_{i,j=o}^{n-1} \xi_{ij} \sin \Omega t_i \cos \Omega t_j \\ &\text{(iii)} \quad A_{22} = \sum_{i,j=o}^{n-1} \xi_{ij} \cos \Omega t_i \cos \Omega t_j \\ &\text{(iv)} \quad D_1 = -\Omega \sum_{i,j=o}^{n-1} \xi_{ij} (V_i - \psi_i') \sin \Omega t_j + \frac{1}{\gamma_3} X_o \\ &\text{(v)} \quad D_2 = \sum_{i,j=o}^{n-1} \xi_{ij} (V_i - \psi_i') \cos \Omega t_j + \frac{1}{\gamma_4} X_o' \end{aligned}$$

The maximum likelihood estimates $\hat{x}(0;T)$, $\widehat{x'}(0;T)$, $\widehat{\mathcal{A}x}(0;T) - B_o$ when the velocity dial readings are given just at times t_i are obtained by solving Eqs. (III.7) for z_1, z_2, z_3. The estimate $\left[\hat{x}(t;T)\right]_n$, $0 \leq t \leq T$, is then obtained by putting the resulting values into Eq. (III.5).

The quantities A_{11}, A_{12}, A_{22}, D_1, D_2 as well as the resulting estimators depend, of course, on n. This has been partially suppressed in the notation for convenience in writing the above equations. Eventually we will evaluate the limiting values of these quantities as $n \to \infty$, for a particular $\phi_V(s,t)$.

The limiting values of all quantities will be denoted by the same symbols as have been used in the above equations. It will always be clear from the context how to interpret the symbols. The limiting values of the

estimates $\hat{x}(0;T)$, $\widehat{x'}(0;T)$, $\widehat{\mathcal{L}x}(0;T) - B_o$ are (see the remark in Section II) the best estimates based on the dial readings for all t, $0 \leq t \leq T$. The best estimate for x(t), for any t in $0 \leq t \leq T$, is gotten by putting the limiting values of $\hat{x}(0;T)$, $\widehat{x'}(0;T)$, and $\widehat{\mathcal{L}x}(0;T)$ into (III.5).*

Adding (iii) to (i) in (III.7) gives

(III.9)
$$z_3 = \frac{\gamma_1(X_o - z_1)}{\gamma_3\Omega^2}$$

or

$$\widehat{\mathcal{L}x}(0;T) - B_o = \frac{\gamma_1(X_o - \hat{x}(0;T))}{\gamma_3\Omega^2}$$

Substituting (III.9) into (III.7) gives

(III.10)
$$\text{(i)} \quad z_1\left[A_{11}\left(1 + \frac{\gamma_1}{\gamma_3\Omega^4}\right) + \frac{1}{\gamma_3}\right] - z_2 A_{12} = D_1 + \frac{A_{11}\gamma_1}{\gamma_3\Omega^4} X_o$$

$$\text{(ii)} \quad -z_1 A_{12}\left(1 + \frac{\gamma_1}{\gamma_3\Omega^4}\right) + z_2\left(A_{22} + \frac{1}{\gamma_4}\right) = D_2 - \frac{A_{12}\gamma_1}{\gamma_3\Omega^4} X_o$$

Some special cases are:

(a) $\begin{cases} \gamma_3 = \infty \text{ (No independent initial estimate of } x_o) \\ \gamma_1 \neq 0 \end{cases}$

---

* Best estimate means best in the sense of the optimization criterion of Section I.D (p.6), with $\hat{x}(T)$ replaced by $\hat{x}(t;T)$ and x(T) replaced by x(t). Cf. the footnote on p. 9.

In this case

$$\text{(III.10a)}\quad \begin{aligned} &\text{(i)} \quad z_1 A_{11} - z_2 A_{12} = D_1 \\ &\text{(ii)} \quad - z_1 A_{12} + z_2 \left(A_{22} + \frac{1}{\gamma_4}\right) = D_2 \end{aligned}$$

$$\text{(b)} \quad \begin{cases} \gamma_1 = 0 & \text{(No accelerometer bias)} \\ \gamma_3 \neq 0 \end{cases}$$

here

$$\text{(III.10b)}\quad \begin{aligned} &\text{(i)} \quad z_1 \left(A_{11} + \frac{1}{\gamma_3}\right) - z_2 A_{12} = D_1 \\ &\text{(ii)} \quad - z_1 A_{12} + z_2 \left(A_{22} + \frac{1}{\gamma_4}\right) = D_2 \end{aligned}$$

The form of $\phi_V(s,t)$ has not been utilized in deriving the above equations, which are therefore valid for any $\phi_V(s,t)$. The form of $\phi_V(s,t)$ affects, of course, the actual values of the quantities A_{11}, A_{12}, A_{22}, D_1, and D_2. Appendix I shows how to evaluate the limits of these quantities as $n \to \infty$, for $\phi_V(s,t)$ of the form $\gamma_2 + \beta_2 e^{-|s - t|\alpha_2}$.

The limits of these quantities for other $\phi_V(s,t)$ can be evaluated by the method of Appendix I--and thus explicit solutions for the optimum position estimate can be obtained--provided one can solve the integral equation (AI.3) with the appropriate $\phi_V(s,t)$ as kernel.

B. EVALUATION OF LIMITING VALUES AS $n \to \infty$ for $\phi_V(s,t) = \gamma_2 + \beta_2 e^{-\alpha_2|s-t|}$

Appendix I shows how to evaluate the limiting values of bilinear forms such as appear in (III.8). The limiting values for the case $\phi_V(s,t) = \gamma_2 + \beta_2 e^{-\alpha_2|s-t|}$ are obtained by applying Eq. (AI.7) of Appendix I, according to the following key:

Quantity evaluated	Values of quantities appearing in Eq. (AI.7)				
	α	β	γ	$u(t)$	$y(t)$
A_{11}	α_2	β_2	γ_2	$\Omega \sin \Omega t$	$\Omega \sin \Omega t$
A_{12}	α_2	β_2	γ_2	$\Omega \sin \Omega t$	$\cos \Omega t$
A_{22}	α_2	β_2	γ_2	$\cos \Omega t$	$\cos \Omega t$
$D_1 - \frac{1}{\gamma_3} X_o$	α_2	β_2	γ_2	$-\left[V(t) - \psi'(t)\right]$	$\Omega \sin \Omega t$
$D_2 - \frac{1}{\gamma_4} X_o'$	α_2	β_2	γ_2	$V(t) - \psi'(t)$	$\cos \Omega t$

The results of this are:

$$A_{11} = \frac{\Omega^2}{2\beta_2}\left\{\frac{\alpha_2 T}{2}\left(1 + \frac{\Omega^2}{\alpha_2^2}\right) - \frac{\alpha_2}{4\Omega}\left(1 - \frac{\Omega^2}{\alpha_2^2}\right)\sin 2\Omega T \right.$$

$$\left. + \sin^2 \Omega T - \frac{\left[\frac{\alpha_2}{\Omega}(1-\cos \Omega T) + \sin \Omega T\right]^2}{2 + \alpha_2 T + \frac{2\beta_2}{\gamma_2}}\right\} \tag{III.11i}$$

(III.11ii)

$$A_{12} = \frac{\Omega}{2\beta_2} \left\{ \frac{\alpha_2}{2\Omega}\left(1 - \frac{\Omega^2}{\alpha_2^{\,2}}\right)\sin^2 \Omega T + \frac{1}{2}\sin 2\Omega T \right.$$

$$\left. - \frac{\left[\frac{\alpha_2}{\Omega}(1 - \cos \Omega T) + \sin \Omega T\right]\left[\frac{\alpha_2}{\Omega}\sin \Omega T + 1 + \cos \Omega T\right]}{2 + \alpha_2 T + \frac{2\beta_2}{\gamma_2}} \right\}$$

(III.11iii)

$$A_{22} = \frac{1}{2\beta_2} \left\{ \frac{\alpha_2 T}{2}\left(1 + \frac{\Omega^2}{\alpha_2^{\,2}}\right) + \frac{\alpha_2}{4\Omega}\left(1 - \frac{\Omega^2}{\alpha_2^{\,2}}\right)\sin 2\Omega T + 1 + \cos^2 \Omega T \right.$$

$$\left. - \frac{\left[\frac{\alpha_2}{\Omega}\sin \Omega T + 1 + \cos \Omega T\right]^2}{2 + \alpha_2 T + \frac{2\beta_2}{\gamma_2}} \right\}$$

(III.11iv)

$$D_1 = \frac{X_o}{\gamma_3} - \frac{\Omega}{2\beta_2} \left\{ \alpha_2\left(1 + \frac{\Omega^2}{\alpha_2^{\,2}}\right)\int_o^T \left[V(t) - \psi'(t)\right] \sin \Omega t \, dt \right.$$

$$- \frac{\Omega}{\alpha_2} V(o) + \left(\sin \Omega T + \frac{\Omega}{\alpha_2}\cos \Omega T\right)\left[V(T) - \psi'(T)\right]$$

$$\left. - \frac{\left[\frac{\alpha_2}{\Omega}(1 - \cos \Omega T) + \sin \Omega T\right]\left[\alpha_2\left(T\,\breve{V}(T) - \psi(T)\right) + V(o) + V(T) - \psi'(T)\right]}{2 + \alpha_2 T + \frac{2\beta_2}{\gamma_2}} \right\}$$

(III.11v)

$$D_2 = \frac{X'_o}{\gamma_4} + \frac{1}{2\beta_2}\left\{ \alpha_2\left(1 + \frac{\Omega^2}{\alpha_2^2}\right)\int_o^T \left[V(t) - \psi'(t)\right] \cos \Omega t \, dt \right.$$

$$+ V(o) + \left[V(T) - \psi'(T)\right]\left(\cos \Omega T - \frac{\Omega}{\alpha_2} \sin \Omega T\right)$$

$$\left. - \frac{\left[\frac{\alpha_2}{\Omega} \sin \Omega T + 1 + \cos \Omega T\right]\left[\alpha_2\left(T\tilde{V}(T) - \psi(T)\right) + V(o) + V(T) - \psi'(T)\right]}{2 + \alpha_2 T + \frac{2\beta_2}{\gamma_2}} \right\}$$

where $\tilde{V}(T) = \frac{1}{T}\int_o^T V(t)\, dt$

The estimators $\widehat{x}(0;T)$, $\widehat{x'}(0;T)$, $\widehat{\alpha x}(0;T) - B_o$ approach limits which are, of course, the solutions to Eqs. (III.9) and (III.10) where the various quantities on the left side now are given by (III.11).

C. <u>ERROR STATISTICS</u>:

Taking mean values on both sides of Eq. (III.10) yields

(III.12)

(i) $$\bar{z}_1\left[A_{11}\left(1 + \frac{\gamma_1}{\gamma_3\Omega^4}\right) + \frac{1}{\beta_3}\right] - \bar{z}_2 A_{12} = x_o\left[A_{11}\left(1 + \frac{\gamma_1}{\gamma_3\Omega^4}\right) + \frac{1}{\gamma_3}\right] - x'_o A_{12}$$

(ii) $$- \bar{z}_1 A_{12}\left(1 + \frac{\gamma_1}{\gamma_3\Omega^4}\right) + \bar{z}_2\left(A_{22} + \frac{1}{\gamma_4}\right) = - x_o A_{12}\left(1 + \frac{\gamma_1}{\gamma_3\Omega^4}\right)$$

$$+ x'_o\left(A_{22} + \frac{1}{\beta_4}\right)$$

Applying these together with Eq. (III.9) gives

$$\overline{z}_1 = x_o \; ; \; \overline{z}_2 = x_o' \; ; \; \overline{z}_3 = 0 \tag{III.13}$$

Thus the estimates of x_o, x_o', and δx_o are unbiased; also, therefore, the estimates of $x(t)$ for any t, $0 \leq t \leq T$. (The property of being unbiased refers, of course, to ensemble averages--i.e., averages over a large number of flights. For any given realization--i.e., single flight--the instrument biases may cause bias in the estimate of $x(t)$.)

Now, for the sake of brevity, put

(III.14)

(i) $K_{11} = A_{11}\left(1 + \frac{\gamma_1}{\gamma_3 \Omega^4}\right) + \frac{1}{\gamma_3}$ (ii) $K_{12} = A_{12}$

(iii) $K_{21} = A_{12}\left(1 + \frac{\gamma_1}{\gamma_3 \Omega^4}\right)$ (iv) $K_{22} = A_{22} + \frac{1}{\gamma_4}$

(v) $E_1 = D_1 + \frac{A_{11}\gamma_1}{\gamma_3 \Omega^4} x_o - \left[D_1 + \frac{A_{11}\gamma_1}{\gamma_3 \Omega^4} x_o\right]$

(vi) $E_2 = D_2 - \frac{A_{12}\gamma_1}{\gamma_3 \Omega^4} x_o - \left[D_2 - \frac{A_{12}\gamma_1}{\gamma_3 \Omega^4} x_o\right]$

and put

$$\delta_1 = z_1 - \overline{z}_1 = \hat{x}(0;T) - x_o \tag{III.15}$$

$$\delta_2 = z_2 - \overline{z}_2 = \widehat{x'}(0;T) - x_o'$$

We know that $\overline{\mathcal{P}_1} = \overline{\mathcal{P}_2} = 0$. Subtracting (III.14) from (III.12) gives

$$\text{(III.16)} \quad \begin{aligned} &\text{(i)} \quad K_{11}\mathcal{P}_1 - K_{12}\mathcal{P}_2 = E_1 \\ &\text{(ii)} \; -K_{21}\mathcal{P}_1 + K_{22}\mathcal{P}_2 = E_2 \end{aligned}$$

Now, subtracting (III.4) from (III.5) gives (for $0 \leqq t \leqq T$)

(III.17)

$$\mathcal{E}_{\hat{x}}(t;T) = \mathcal{P}_1 \cos \Omega t + \mathcal{P}_2 \frac{\sin \Omega t}{\Omega} + \frac{\hat{\mathcal{A}}x(0;T) - \mathcal{A}x_o}{\Omega^2}(1 - \cos \Omega t)$$

In view of (III.9), this may be written

$$\text{(III.17')} \quad \mathcal{E}_{\hat{x}}(t;T) = \mathcal{P}_1\left[\cos \Omega t - \frac{\gamma_1}{\gamma_3 \Omega^4}(1 - \cos \Omega t)\right] + \mathcal{P}_2 \frac{\sin \Omega t}{\Omega} + \mathcal{E}_{X_o}\left[\frac{\gamma_1}{\gamma_3 \Omega^4}(1 - \cos \Omega t)\right] + \frac{\mathcal{E}_B}{\Omega^2}(1 - \cos \Omega t)$$

Thus, to obtain $\overline{\mathcal{E}^2_{\hat{x}}(t;T)}$, it suffices to know $\overline{\mathcal{P}_1^2}$, $\overline{\mathcal{P}_2^2}$, $\overline{\mathcal{P}_1\mathcal{P}_2}$, $\overline{\mathcal{P}_1\mathcal{E}_{X_o}}$, $\overline{\mathcal{P}_1\mathcal{E}_B}$, $\overline{\mathcal{P}_2\mathcal{E}_B}$. From (III.16), it is seen that it suffices to know

$\overline{E_1^2}$, $\overline{E_2^2}$, $\overline{E_1E_2}$, $\overline{E_1\mathcal{E}_{X_o}}$, $\overline{E_2\mathcal{E}_{X_o}}$, $\overline{E_1\mathcal{E}_B}$, $\overline{E_2\mathcal{E}_B}$.

Now

$$E_1 = -\Omega \sum_{i,j=o}^{n-1} \xi_{ij} \left[\mathcal{E}_V(t_i) - \frac{\sin \Omega t_i}{\Omega} \mathcal{E}_B \right] \sin \Omega t_j$$

$$+ \mathcal{E}_{X_o} \left(1 + \frac{A_{11} \gamma_1}{\Omega^4} \right) \frac{1}{\gamma_3}$$

and

$$E_2 = \sum_{i,j=o}^{n-1} \xi_{ij} \left[\mathcal{E}_V(t_i) - \frac{\sin \Omega t_i}{\Omega} \mathcal{E}_B \right] \cos \Omega t_j$$

$$+ \frac{1}{\gamma_4} \mathcal{E}_{X_o'} - \frac{A_{12} \gamma_1}{\gamma_3 \Omega^4} \mathcal{E}_{X_o}$$

or

$$\text{(i)} \quad E_1 = -\Omega \sum_{i,j=o}^{n-1} \xi_{ij} \mathcal{E}_V (t_i) \sin \Omega t_j$$

$$\text{(III.18)} \qquad + \frac{A_{11}}{\Omega^2} \mathcal{E}_B + \mathcal{E}_{X_o} \left(1 + \frac{A_{11} \gamma_1}{\Omega^4} \right) \frac{1}{\gamma_3}$$

$$\text{(ii)} \quad E_2 = \sum_{i,j=o}^{n-1} \xi_{ij} \mathcal{E}_V (t_i) \cos \Omega t_j$$

$$- \frac{A_{12}}{\Omega^2} \mathcal{E}_B - \frac{A_{12} \gamma_1}{\gamma_3 \Omega^4} \mathcal{E}_{X_o} + \frac{1}{\gamma_4} \mathcal{E}_{X_o'}$$

The evaluation of the quantities $\overline{E_1^2}$, etc., can be illustrated by the evaluation of $\overline{E_1^2}$:

(III.19)
$$\overline{E_1^{\,2}} = \Omega^2 \sum_{ijk\ell} \xi_{ij}\, \xi_{k\ell}\, \overline{\mathcal{E}_V(t_i)\ \mathcal{E}_V(t_k)}\, \sin \Omega t_j \sin \Omega t_\ell$$

$$+ \frac{A^2_{11}}{\Omega^4}\, \overline{\mathcal{E}^2_B} + \frac{1}{\gamma_3^{\,2}} \left(1 + \frac{A_{11}\,\gamma_1}{\Omega^4}\right)^2 \overline{\mathcal{E}^2_{X_o}}$$

Using the fact that $\overline{\mathcal{E}_V(t_i)\ \mathcal{E}_V(t_k)} = \phi_V(t_i, t_k)$ and $\sum_{k=o}^{n-1} \phi_{ik}\, \xi_{k\ell} = \delta_{i\ell}$; and also $\overline{\mathcal{E}^2_B} = \gamma_1$, $\overline{\mathcal{E}^2_{X_o}} = \gamma_3$ one can easily evaluate this expression. The result, together with the results for the other necessary evaluations, is:

(III.20) (i) $$\overline{E_1^2} = \left(1 + \frac{A_{11}\,\gamma_1}{\Omega^4}\right)\left[A_{11} + \frac{1}{\gamma_3}\left(1 + \frac{A_{11}\,\gamma_1}{\Omega^4}\right)\right]$$

(ii) $$\overline{E_2^2} = A_{22} + \frac{1}{\gamma_4} + \frac{A^2_{12}\,\gamma_1}{\Omega^4}\left(1 + \frac{\gamma_1}{\gamma_3 \Omega^4}\right)$$

(iii) $$\overline{E_1 E_2} = -\, A_{12}\left(1 + \frac{\gamma_1}{\gamma_3 \Omega^4}\right)\left(1 + \frac{\gamma_1\, A_{11}}{\Omega^4}\right)$$

(iv) $$\overline{E_1\, \mathcal{E}_{X_o}} = \left(1 + \frac{A_{11}\,\gamma_1}{\Omega^4}\right)$$

(v) $$\overline{E_2\, \mathcal{E}_{X_o}} = \frac{-A_{12}\,\gamma_1}{\Omega^4}$$

(vi) $$\overline{E_1\, \mathcal{E}_B} = \frac{A_{11}\,\gamma_1}{\Omega^2}$$

(vii) $$\overline{E_2\, \mathcal{E}_B} = \frac{-A_{12}\,\gamma_1}{\Omega^2}$$

The equations derived in this section for mean values and variances of the various estimators hold in the limit as $n \to \infty$ (with, of course, the quantities A_{11}, etc., taking on their limiting values as $n \to \infty$).

All the relations (III.12) - (III.20) hold for any autocorrelation function $\phi_V(s,t)$. As noted above (p. 17) the form of $\phi_V(s,t)$ determines the actual values of A_{11}, etc.

The calculation of $\overline{\mathcal{E}^2_{\hat{x}}(t;T)}$ is now a straightforward, though very cumbersome, application of the results given by Eqs. (III.16), (III.17'), and (III.20). Because of the cumbersome nature of the resulting formula, it is not thought worthwhile actually to write it out here. However, certain conclusions can be drawn fairly easily. For example, one may use these equations together with Eqs. (III.11) to calculate the behavior of the estimator variances as $T \to \infty$, for $\phi_V(s,t) = \gamma_2 + \beta_2 e^{-\alpha_2|s-t|}$.

The results are: the error variance, for sufficiently large T, approaches the quantity $\dfrac{\gamma_1\gamma_3}{\gamma_1 + \gamma_3\Omega^4}$. (This checks a more general result obtained in Section VI, namely that this quantity is the asymptotic error variance in Case 1 for any ϕ_V such that $\mathcal{E}_V(t)$ is stationary and has a power spectrum containing no delta function at frequency Ω.) For γ_1 or $\gamma_3 = 0$, the error variance goes to zero as $\frac{1}{T}$ for $\phi_V(s,t) = \gamma_2 + \beta_2 e^{-\alpha_2|s-t|}$.

D. SUMMARY OF RESULTS, CASE 1

The best estimate of position at time T, based on all information received up to time T, is

$$\text{(III.21)} \quad \hat{x}(T) = \int_0^T B(\tau)\,\frac{\sin\Omega(T-\tau)}{\Omega}\,d\tau + \frac{\widehat{\mathcal{L}x}(0;T) - B_0}{\Omega^2}\,(1-\cos\Omega t)$$

$$+ \hat{x}(0;T)\cos\Omega T + \hat{x}'(0;T)\,\frac{\sin\Omega T}{\Omega}$$

where $\hat{x}(0;T)$, $\widehat{x'}(0;T)$, $B_o - \widehat{\mathcal{L}x}(0;T)$ are respectively the best estimates of initial position, initial velocity, and accelerometer bias, based on all information up to time T. $\hat{x}(0;T)$, $\widehat{x'}(0;T)$, and $\widehat{\mathcal{L}x}(0;T) - B_o$ are solutions of a set of simultaneous linear equations (Eqs. (III.9), (III.10), with $z_1 = \hat{x}(0;T)$; $z_2 = \widehat{x'}(0;T)$; and $z_3 = \widehat{\mathcal{L}x}(0;T) - B_o$).

This holds for any velocity noise autocorrelation function $\phi_V(s,t)$. The function $\phi_V(s,t)$ affects the values of the coefficients of the simultaneous equations. For $\phi_V(s,t) = \gamma_2 + \beta_2 e^{-a_2|s-t|}$, these values can be found from Eqs. (III.11).

In other words, $\hat{x}(T)$ is found by solving the accelerometer equation (Eq. (I.1)) with B(t), the accelerometer dial reading, as driving function; subtracting a term which represents the best estimate of the effect of accelerometer bias; and using as initial conditions the best estimates (up to time T) of initial position and velocity.

As elapsed time T grows larger, the computer continuously re-evaluates $\hat{x}(0;T)$, $\widehat{x'}(0;T)$, and $\widehat{\mathcal{L}x}(0;T) - B_o$. (For $0 \leq t < T$, one can get the best estimate of what the position was at time t from Eq. (III.5). Thus, for any fixed $t < T$, the computer can be used to get a continually better estimate of x(t) as T grows larger.)

The estimates $\hat{x}(0;T)$, $\widehat{x'}(0;T)$, and $\widehat{\mathcal{L}x}(0;T) - B_o$ have errors which have zero mean (over a large number of flights), are linear functions of the dial errors, and do not depend on the path. Thus

$$\text{(III.22)} \qquad \hat{x}(T) = x(T) + \mathcal{E}_{\hat{x}(T)}$$

where $\mathcal{E}_{\hat{x}(T)}$ has zero mean, is a linear function of the dial errors, and is independent of the path. The variance of $\mathcal{E}_{\hat{x}(T)}$ can be found from Eqs. (III.16), (III.17'), and (III.20)

IV. TREATMENT OF CASE 2: $\phi_V(s,t) = \gamma_2$

A. DERIVATION OF OPTIMUM POSITION ESTIMATE

In this case, all sources of error are present except for the non-bias component of accelerometer noise.

Since $x'(t) = v(t)$, we have

(IV.1) $$x(t) = \int_0^t v(\tau)\, d\tau + x_o$$

Also in this case $\mathcal{E}_V(t) = V(0) - v(0)$, so that

(IV.2) $$V(t) = v(t) + V_o - x_o'$$

Let

(IV.3) $$W(t) = \int_0^t V(\tau)\, d\tau$$

then, putting (IV.2) into (IV.1),

(IV.4) $$x(t) = W(t) + x_o + (x_o' - V_o)\, t$$

and $$\mathcal{L}x(t) = \mathcal{L}W(t) + \Omega^2 x_o + (x_o' - V_o)\, \Omega^2 t$$

Therefore

(IV.5) $$\hat{x}(t;T) = W(t) + \hat{x}(0;T) + (\widehat{x'}(0;T) - V_o)\, t$$

$$\widehat{\mathcal{L}x}(t;T) = \mathcal{L}W(t) + \Omega^2 \hat{x}(0;T) + (\widehat{x'}(0;T) - V_o)\, \Omega^2 t$$

The quantity which must be minimized with respect to $\hat{x}(0;T)$ and $\widehat{x'}(0;T)$ is

$$Q_n = \sum_{i,j=0}^{n-1} \eta_{ij} (B_i - \widehat{\mathcal{L}x_i})(B_j - \widehat{\mathcal{L}x_j}) + \frac{1}{\gamma_3}\left[X_o - \hat{x}(0;T)\right]^2 + \frac{1}{\gamma_4}\left[X_o' - \widehat{x'}(0;T)\right]^2 + \frac{1}{\gamma_2}\left[V_o - \widehat{x'}(0;T)\right]^2 \tag{IV.6}$$

where now $\left[\eta_{ij}\right] = \left[\phi_B(t_i,t_j)\right]^{-1}$ (matrix inverse)

Q_n is, apart from a constant independent of $\hat{x}(0;T)$ and $\widehat{x'}(0;T)$, equal to $-\ln p\left[U \middle| \hat{x}(0;T), \widehat{x'}(0;T)\right]$.

Let $z_1 = \hat{x}(0;T)$, $z_2 = \widehat{x'}(0;T) - V_o$

Differentiating Q_n with respect to z_1 and z_2 after putting (IV.5) into (IV.6) and equating the derivatives to zero gives

$$\text{(i)}\quad z_1\left(A_{11} + \frac{1}{\gamma_3}\right) + z_2 A_{12} = D_1$$

$$\text{(ii)}\quad z_1 A_{12} + z_2\left(A_{22} + \frac{1}{\gamma_2} + \frac{1}{\gamma_4}\right) = D_2 \tag{IV.7}$$

where A_{11}, A_{12}, A_{22}, D_1, and D_2 are given by

$$
\text{(IV.8)} \quad \begin{cases}
A_{11} = \Omega^4 \sum\limits_{i,j=o}^{n-1} \eta_{ij} \\[2ex]
A_{12} = \Omega^4 \sum\limits_{i,j=o}^{n-1} \eta_{ij}\, t_i \\[2ex]
A_{22} = \Omega^4 \sum\limits_{i,j=o}^{n-1} \eta_{ij}\, t_i\, t_j \\[2ex]
D_1 = \Omega^2 \sum\limits_{i,j=o}^{n-1} \eta_{ij}\, (B_i - \rho_1 W_i) + \dfrac{X_o}{\gamma_3} \\[2ex]
D_2 = \Omega^2 \sum\limits_{i,j=o}^{n-1} \eta_{ij}\, (B_i - \rho_1 W_i)\, t_j + \dfrac{X_o' - V_o}{\gamma_4}
\end{cases}
$$

The best estimate of $x(t)$, $0 \leqq t \leqq T$, when the accelerometer dial readings are given just at times t_i is now obtained by putting the solutions to (IV.7) into (IV.5). As in Case 1, we will evaluate the limiting values of the solutions z_1, z_2 of (IV.7); the best estimate of $x(t)$, $0 \leqq t \leqq T$, will then be obtained by putting these limiting values into (IV.5).

The above equations are valid for any $\phi_B(s,t)$. The method of Appendix I can be applied to evaluate the limits of A_{11}, A_{12}, A_{22}, D_1, and D_2 for $\phi_B(s,t) = \gamma_1 + \beta_1 e^{-|s-t|\alpha_1}$.

The limits of these quantities for other $\phi_B(s,t)$ can be evaluated by the method of Appendix I provided one can solve the integral Eq. (AI.3) with the appropriate $\phi_B(s,t)$ as kernel.

B. LIMITING VALUES AS $n \to \infty$ for $\phi_B(s,t) = \gamma_1 + \beta_1 e^{-\alpha_1 |s-t|}$

One applies Eq. (AI.7) according to the following key:

	Values of quantities appearing in Eq. (AI.7)				
Quantity evaluated	α	β	γ	$u(t)$	$y(t)$
A_{11}	α_1	β_1	γ_1	Ω^2	Ω^2
A_{12}	α_1	β_1	γ_1	$\Omega^2 t$	Ω^2
A_{22}	α_1	β_1	γ_1	$\Omega^2 t$	$\Omega^2 t$
$D_1 - \frac{X_o}{\gamma_3}$	α_1	β_1	γ_1	$B(t) - \alpha^F W(t)$	Ω^2
$D_2 - \frac{X_o' - V_o}{\gamma_4}$	α_1	β_1	γ_1	$B(t) - \alpha^P W(t)$	$\Omega^2 t$

The results are:

$$A_{11} = \frac{\Omega^4 (2+\alpha_1 T)}{2\beta_1} \left[1 - \frac{1}{1 + \frac{2\beta_1}{\gamma_1 (2+\alpha_1 T)}} \right] \tag{IV.9i}$$

$$A_{12} = \frac{\Omega^4 (2+\alpha_1 T)\, T}{4\beta_1} \left[1 - \frac{1}{1 + \frac{2\beta_1}{\gamma_1 (2+\alpha_1 T)}} \right] \tag{IV.9ii}$$

$$A_{22} = \frac{\Omega^4 T^2}{2\beta_1} \left[1 + \frac{\alpha_1 T}{3} + \frac{1}{\alpha_1 T} - \frac{\frac{1}{4}(2+\alpha_1 T)}{1 + \frac{2\beta_1}{\gamma_1 (2+\alpha_1 T)}} \right] \tag{IV.9iii}$$

(IV.9iv)

$$D_1 = \frac{\Omega^2}{2\beta_1}\left[\alpha_1 T\,\widetilde{(B - \alpha W)} + B_o - \alpha W_o + B(T) - \alpha W(T)\right]\left[1 - \frac{1}{1 + \frac{2\beta_1}{\gamma_1(2+\alpha_1 T)}}\right] + \frac{X_o}{\gamma_3}$$

(IV.9v)

$$D_2 = \frac{\Omega^2}{2\beta_1}\left\{\alpha_1\int_o^T t\left[B(t) - \alpha W(t)\right] dt - \frac{1}{\alpha_1}(B_o - \alpha W_o) + (T + \frac{1}{\alpha_1})\left[B(T) - \alpha W(T)\right]\right.$$

$$\left. - \frac{\frac{T}{2}\left[\alpha_1 T\,\widetilde{(B - \alpha W)} + B_o - \alpha W_o + B(T) - \alpha W(T)\right]}{1 + \frac{2\beta_1}{\alpha_1(2+\alpha_1 T)}}\right\} + \frac{X_o' - V_o}{\gamma_4}$$

where $$\widetilde{B - \alpha W} = \frac{1}{T}\int_o^T \left[B(t) - \alpha W(t)\right] dt$$

C. <u>ERROR STATISTICS</u>:

Taking mean values on both sides of (IV.7) gives

(i) $$\bar{z}_1 (A_{11} + \frac{1}{\gamma_3}) + \bar{z}_2 A_{12} = (A_{11} + \frac{1}{\gamma_3}) x_o$$

(IV.10)

(ii) $$\bar{z}_1 A_{12} + \bar{z}_2 (A_{22} + \frac{1}{\gamma_2} + \frac{1}{\gamma_4}) = A_{12} x_o$$

Applying these equations gives

(IV.11) $$\bar{z}_1 = x_o \; ; \; \bar{z}_2 = 0$$

Putting

$$\rho_1 = z_1 - \bar{z}_1 = \hat{x}(0;T) - x_o \tag{IV.12}$$

$$\rho_2 = z_2 - \bar{z}_2 = \widehat{x'}(0;T) - V_o$$

and

$$E_1 = D_1 - \bar{D}_1 \tag{IV.13}$$

$$E_2 = D_2 - \bar{D}_2$$

gives

(IV.14) (i) $\rho_1 \left(A_{11} + \frac{1}{\gamma_3}\right) + \rho_2 A_{12} = E_1$

(ii) $\rho_1 A_{12} + \rho_2 \left(A_{22} + \frac{1}{\gamma_2} + \frac{1}{\gamma_4}\right) = E_2$

Now, subtracting (IV.4) from (IV.5) gives

$$\mathcal{E}_{\hat{x}}(t;T) = (\hat{x}(0;T) - x_o) + (\widehat{x'}(0;T) - x_o')\, t \tag{IV.15}$$

$$= \rho_1 + (\rho_2 + \mathcal{E}_V)\, t$$

Therefore, to obtain $\overline{\mathcal{E}^2_{\hat{x}}(t;T)}$, one needs to know

$\overline{\rho_1^{\,2}}$, $\overline{\rho_2^{\,2}}$, $\overline{\rho_1 \rho_2}$, $\overline{\rho_1 \mathcal{E}_V}$, $\overline{\rho_2 \mathcal{E}_V}$. From Eq. (IV.14) it is evident that it suffices to know $\overline{E_1^{\,2}}$, $\overline{E_2^{\,2}}$, $\overline{E_1 E_2}$, $\overline{E_1 \mathcal{E}_V}$, $\overline{E_2 \mathcal{E}_V}$.

Now

$$\text{(i)}\quad E_1 = \Omega^2 \sum_{i,j=o}^{n-1} \eta_{ij} \left[\mathcal{E}_B(t_i) - \Omega^2 t_i \ \mathcal{E}_V \right] + \frac{\mathcal{E}_{X_o}}{\gamma_3}$$

$$\text{(IV.16)}\qquad = \Omega^2 \sum_{i,j=o}^{n-1} \eta_{ij} \ \mathcal{E}_B(t_i) - \mathcal{E}_V A_{12} + \frac{\mathcal{E}_{X_o}}{\gamma_3}$$

$$\text{(ii)}\quad E_2 = \Omega^2 \sum_{i,j=o}^{n-1} \eta_{ij} \ \mathcal{E}_B(t_i) \ t_j - (A_{22} + \frac{1}{\gamma_4}) \ \mathcal{E}_V + \frac{\mathcal{E}_{X'_o}}{\gamma_4}$$

Evaluating the necessary quantities in a manner similar to that used in Case 1, one obtains:

$$\text{(i)}\quad \overline{E_1^{\,2}} = A_{11} + \frac{1}{\gamma_3} + \gamma_2 A_{12}^{\,2}$$

$$\text{(ii)}\quad \overline{E_2^{\,2}} = A_{22} + \frac{1}{\gamma_4} + \gamma_2 \left(A_{22} + \frac{1}{\gamma_4} \right)^2$$

$$\text{(IV.17)}\qquad \text{(iii)}\quad \overline{E_1 E_2} = A_{12} \left[1 + \frac{\gamma_2}{\gamma_4} + \gamma_2 A_{22} \right]$$

$$\text{(iv)}\quad \overline{E_1 \mathcal{E}_V} = - \gamma_2 A_{12}$$

$$\text{(v)}\quad \overline{E_2 \mathcal{E}_V} = - \gamma_2 \left(A_{22} + \frac{1}{\gamma_4} \right)$$

All the relations (IV.10)-(IV.17) hold for any autocorrelation function $\phi_B(s,t)$. The form of $\phi_B(s,t)$ determines the actual values of A_{11}, etc. All these relations hold in the limit as $n \to \infty$.

The error variance can be calculated in a straightforward manner using the above equations. As in Case 1, the resulting formula will not be stated because of its cumbersome nature.

Applying the above equations together with (IV.9) for $\phi_B(s,t) = \gamma_1 + \beta_1 e^{-\alpha_1|s-t|}$, one obtains the result that the asymptotic error variance for large T is $\dfrac{\gamma_1\gamma_3}{\gamma_1 + \gamma_3\Omega^4}$. (This checks a more general result obtained in Section VI, namely, that this quantity is the asymptotic error variance in Case 2 for any ϕ_B such that $\mathcal{E}_B(t)$ is stationary.) For γ_1 or $\gamma_3 = 0$, the error variance goes to zero as $\frac{1}{T}$ for $\phi_B(s,t) = \gamma_1 + \beta_1 e^{-\alpha_1|s-t|}$.

D. SUMMARY OF RESULTS, CASE 2

The best estimate of position at time T, based on all information received up to time T, is

$$\hat{x}(T) = \int_0^T V(\tau)d\tau + \hat{x}(0;T) + (\widehat{x'}(0;T) - V_0)T \tag{IV.18}$$

where $\hat{x}(0;T)$ and $V_0 - \widehat{x'}(0;T)$ are respectively the best estimates of initial position and of velocity dial bias, based on all information up to time T. $\hat{x}(0;T)$ and $\widehat{x'}(0;T) - V_0$ are solutions of a pair of linear simultaneous equations (Eqs. (IV.7), with $z_1 = \hat{x}(0;T)$; $z_2 = \widehat{x'}(0;T) - V_0$).

This holds for any accelerometer noise autocorrelation function $\phi_B(s,t)$. The function $\phi_B(s,t)$ affects the values of the coefficients of the simultaneous equations. For $\phi_B(s,t) = \gamma_1 + \beta_1 e^{-\alpha_1|s-t|}$, these values can be found from Eq. (IV.9).

Thus, $\hat{x}(T)$ is formed by integrating the velocity dial reading V(t); subtracting a term which represents the best estimate of the effect of

velocity dial bias; and using as initial position the best estimate of x_0 based on all information received up to time T.

The estimates $\hat{x}(0;T)$ and $\widehat{x'}(0;T)$ have errors which have zero mean, are linear functions of the instrument errors, and do not depend on the true path. The same things therefore are true of the error $\mathcal{E}_{\hat{x}(T)}$:

$$\hat{x}(T) = x(T) + \mathcal{E}_{\hat{x}(T)} \tag{IV.19}$$

where $\mathcal{E}_{\hat{x}(T)}$ has mean zero, is a linear function of instrument errors, and is independent of the true path $x(t)$. The variance of $\mathcal{E}_{\hat{x}(T)}$ can be found from Eqs. (IV.14), (IV.15), and (IV.17).

V. OPTIMUM POSITION ESTIMATION AS A FILTERING PROBLEM

A. REDUCTION OF THE FOUR DIAL READINGS TO TWO DIAL READINGS

It is possible to reduce the four dial readings $B(t)$, $V(t)$, X_o, X_o' to two dial readings without loss of information.

Let

$$(V.1)\quad \begin{cases} Q(t) = X_o + \int_o^t V(\tau)\, d\tau \\ R(t) = X_o \cos \Omega t + \dfrac{X_o'}{\Omega} \sin \Omega t + \dfrac{1}{\Omega} \int_o^t B(\tau) \sin \Omega (t - \tau)\, d\tau \\ S(t) = Q(t) - R(t) \end{cases}$$

The original dial readings can be obtained from a knowledge of any two of the dials Q, R, S. For example, if one knows R and S:

$$(V.2)\quad \begin{cases} X_o = R(0) + S(0) \\ X_o' = R'(0) \\ V(t) = R'(t) + S'(t) \\ B(t) = R''(t) + \Omega^2 R(t) \end{cases}$$

Thus the problem of obtaining an estimate $\hat{x}(t)$ for $x(t)$ from a knowledge of $B(t)$, $V(t)$, X_o, X_o' is completely equivalent to the problem of obtaining $\hat{x}(t)$ from a knowledge of any two of the three dials $R(t)$, $Q(t)$, $S(t)$.

Note that $Q(t)$ and $R(t)$ each are estimates of $x(t)$; $S(t)$ is the difference between these two estimates of $x(t)$.

The advantage of the reduction from four dials to two is that it is simpler to visualize intuitively what the optimum process amounts to if one considers the input dials to be, say, $R(t)$ and $S(t)$, rather than the original four dials. Also, for the purpose of calculating asymptotic errors (see Section VI), the reduction turns out to be mathematically convenient.

If we define

$$(V.3)\quad \begin{cases} \mathcal{E}_Q(t) = \mathcal{E}_{X_o} + \int_o^t \mathcal{E}_B(\tau)\, d\tau \\ \mathcal{E}_R(t) = \mathcal{E}_{X_o} \cos \Omega t + \dfrac{\mathcal{E}_{X'_o}}{\Omega} \sin \Omega t + \dfrac{1}{\Omega}\int_o^t \mathcal{E}_B(\tau) \sin \Omega(t-\tau)\, d\tau \\ \mathcal{E}_S(t) = \mathcal{E}_Q(t) - \mathcal{E}_R(t) \end{cases}$$

we are led to the result

$$(V.4)\quad \begin{cases} R(t) = x(t) + \mathcal{E}_R(t) \\ Q(t) = x(t) + \mathcal{E}_Q(t) \\ S(t) = \mathcal{E}_S(t) = \mathcal{E}_Q(t) - \mathcal{E}_R(t) \end{cases}$$

The problem assumes a particularly convenient form if one considers the pair of dials $\left[R(t), S(t)\right]$ or the pair $\left[Q(t), S(t)\right]$. For expository purposes, we will consider the pair $\left[R(t), S(t)\right]$. These will be referred to as eigendials.

From (V.4) it is apparent that an estimate of $x(T)$ can be obtained by estimating $\mathcal{E}_R(T)$ on the basis of the readings $R(t)$, $S(t)$ for $0 \leq t \leq T$, and then subtracting this from $R(T)$. In other words, one can define an estimate of $x(T)$ by

$$\text{(V.5)} \qquad \hat{x}(T) = R(T) - \hat{\mathcal{E}}_R(T)$$

where $\hat{\mathcal{E}}_R(T)$ denotes the estimate of $\mathcal{E}_R(T)$ based on $R(t)$, $S(t)$ for $0 \leq t \leq T$.

From this it follows that (subtracting (V.4) from (V.5))

$$\text{(V.6)} \qquad \text{Error in } \hat{x}(T) = \text{Error in } \hat{\mathcal{E}}_R(T)$$

It is also true that all possible estimates of $x(T)$ in terms of $R(t)$, $S(t)$ for $0 \leq t \leq T$ are of the form (V.5) for suitably defined $\hat{\mathcal{E}}_R(T)$, since, given an estimate $\hat{x}(T)$, one can define the estimate of $\mathcal{E}_R(T)$ by $\hat{\mathcal{E}}_R(T) = R(T) - \hat{x}(T)$.

We can conclude that any process of estimation of $x(T)$ on the basis of the dial readings for $0 \leq t \leq T$ is equivalent to first estimating $\mathcal{E}_R(T)$ on the basis of the eigendial readings for $0 \leq t \leq T$ and then obtaining $\hat{x}(T)$ by (V.5).*

B. <u>REDUCTION TO A FILTERING PROBLEM</u>

According to what was said in the introduction, one desires to obtain a linear estimate of $x(T)$ such that $\overline{\left[\hat{x}(T) - x(T)\right]^2}$ is a minimum, and this estimate must not utilize any a-priori information about the possible paths $x(t)$. In view of (V.5) and (V.6) the first requirement means that one desires to find an estimate $\hat{\mathcal{E}}_R(T)$ of $\mathcal{E}_R(T)$, formed by a linear operation on $R(t)$, $S(t)$ for $0 \leq t \leq T$, for which $\overline{\left[\hat{\mathcal{E}}_R(T) - \mathcal{E}_R(T)\right]^2}$ is a minimum. The second requirement obviously implies that $R(t)$ yields no information about $\mathcal{E}_R(t)$ - i.e.

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*This is closely related to a more specialized result of J. H. Laning; see Ref. 5.

that the estimate of $\mathcal{E}_R(T)$ must be based on S(t) alone. In short, the problem amounts to one of linear least squares filtering of
$\mathcal{E}_R(t)$ from $S(t) = \mathcal{E}_Q(t) - \mathcal{E}_R(t)$.*

C. ALTERNATE DEFINITION OF EIGENDIALS

For some purposes it may be more convenient to adapt Q(t), S(t) as eigendials. In this case, one must find a linear least-squares estimate of $\mathcal{E}_Q(T)$ base on S(t), $0 \leqq t \leqq T$. The equation for the position estimate is now

$$\hat{x}(T) = Q(T) - \hat{\mathcal{E}}_Q(T) \tag{V.7}$$

Also

$$\text{Error in } \hat{x}(T) = \text{Error in } \hat{\mathcal{E}}_Q(T) \tag{V.8}$$

For other purposes it might be best to use R(t) and Q(t) as eigendials. This might be the case if one wished to apply the maximum likelihood method. (See p. 12)

It should be understood, of course, that these various possibilities are merely tantamount to alternative mathematical notations, and have no effect on the final answer.

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*If it were desired to take advantage of a-priori information about x(t) then $\hat{\mathcal{E}}_R(T)$ would have to be based on both R(t) and S(t).

VI. ASYMPTOTIC VARIANCE OF THE POSITION ESTIMATE

A. BACKGROUND

For some guidance applications it is of interest to consider the variance of $\hat{x}(T)$ for values of T large compared to $1/\Omega$. This section is devoted to a discussion of this subject.

It is convenient to distinguish between the bias and non-bias components of the dial errors:

Let

$$\text{(VI.1)} \qquad \begin{aligned} K_V &= \text{Velocity dial bias;} \quad \tilde{\mathcal{E}}_V(t) = \mathcal{E}_V(t) - K_V \\ K_B &= \text{Accelerometer bias;} \quad \tilde{\mathcal{E}}_B(t) = \mathcal{E}_B(t) - K_B \end{aligned}$$

According to the preceding section, the estimation of x(T) is equivalent to estimating $\mathcal{E}_Q(T)$ by operating on S(t), $0 \leq t \leq T$, and then setting $\hat{x}(T) = Q(T) - \hat{\mathcal{E}}_Q(T)$. The error in $\hat{x}(T)$ then equals the error in $\hat{\mathcal{E}}_Q(T)$. The asymptotic variance of $\hat{x}(T)$ will depend on the accuracy with which

$$\text{(VI.2)} \qquad \mathcal{E}_Q(t) = \mathcal{E}_{X_0} + K_V t + \int_0^t \tilde{\mathcal{E}}_V(\tau)\, d\tau$$

can be filtered from

$$\text{(VI.3)} \qquad S(t) = \left(\mathcal{E}_{X_0} - \frac{K_B}{\Omega^2}\right)(1-\cos\Omega t) + K_V t + \int_0^t \tilde{\mathcal{E}}_V(\tau)\, d\tau$$

$$- \frac{\mathcal{E}_{X_0'}}{\Omega} \sin \Omega t - \frac{1}{\Omega}\int_0^t \tilde{\mathcal{E}}_B(\tau) \sin \Omega(t-\tau)\, d\tau$$

It will be assumed throughout this section that $\mathcal{E}_V(t)$ and $\mathcal{E}_B(t)$ are stationary processes, possessing autocorrelation functions.

B. CASE WHERE $\tilde{\mathcal{E}}_B(t) = 0$

A process of operating on S(t) will be described, by which all terms of (VI.3) except for $\mathcal{E}_{X_o}$ can be extracted exactly from S(t) as $t \to \infty$; and by which the problem of estimating $\mathcal{E}_Q(T)$ can be reduced to the trivial problem of estimating $\mathcal{E}_{X_o}$ from a knowledge of $\mathcal{E}_{X_o} - \frac{K_B}{\Omega^2}$.

Now,

$$S'(t) = K_V + \tilde{\mathcal{E}}_V(t) + \left(\mathcal{E}_{X_o} - \frac{K_B}{\Omega^2}\right)\Omega \sin \Omega t \tag{VI.4}$$

$$- \mathcal{E}_{X'_o} \cos \Omega t$$

From (VI.3), K_V can be obtained by

$$K_V = \lim_{T \to \infty} \frac{S(T)}{T} \tag{VI.5}$$

Also, assuming the spectrum of $\tilde{\mathcal{E}}_V(t)$ has no delta function at frequency Ω, one has, by (VI.4),

$$\mathcal{E}_{X_o} - \frac{K_B}{\Omega^2} = 2 \lim_{T \to \infty} \frac{1}{T} \int_0^T \left[S'(t) - K_V\right] \frac{\sin \Omega t}{\Omega} \, dt \tag{VI.6}$$

$$\frac{\mathcal{E}_{X'_o}}{\Omega} = -2 \lim_{T \to \infty} \frac{1}{T} \int_0^T \left[S'(t) - K_V\right] \cos \Omega t \, dt \tag{VI.7}$$

$$\int_0^t \mathcal{E}_V(\tau)d\tau = S(t) - \left(\mathcal{E}_{X_o} - \frac{K_B}{\Omega^2}\right)(1 - \cos\Omega t) - K_V t + \frac{\mathcal{E}_{X'_o}}{\Omega}\sin\Omega t \tag{VI.8}$$

From a knowledge of $S(t)$ as $t \to \infty$ we have thus obtained the exact value of each of the terms on the right side of (VI.3), including the value of $\mathcal{E}_{X_o} - \frac{K_B}{\Omega^2}$. It is clear from (VI.3) that this is the most information regarding $\mathcal{E}_{X_o}$ that can be extracted from $S(t)$.

We have also obtained the exact value of the quantities on the right side of (VI.2) with the exception of $\mathcal{E}_{X_o}$.

The conclusion is: the asymptotic variance of $\hat{x}(T)$ can be made as small, but no smaller, than the variance with which $\mathcal{E}_{X_o}$ can be estimated from $\mathcal{E}_{X_o} - \frac{K_B}{\Omega^2}$.

(Note that we have made few assumptions about the statistics of $\mathcal{E}_V(t)$.)

If $\overline{\mathcal{E}^2_{X_o}} = \gamma_3$, $\overline{K^2_B} = \gamma_1$, the least-squares linear estimate of $\mathcal{E}_{X_o}$ on the basis of $\mathcal{E}_{X_o} - \frac{K_B}{\Omega^2}$ is easily shown to be

$$\hat{\mathcal{E}}_{X_o} = \frac{\gamma_3}{\gamma_3 + \frac{\gamma_1}{\Omega^4}}\left(\mathcal{E}_{X_o} - \frac{K_B}{\Omega^4}\right) \tag{VI.9}$$

The variance of this is

$$\overline{\left[\hat{\mathcal{E}}_{X_o} - \mathcal{E}_{X_o}\right]^2} = \frac{\gamma_1\gamma_3}{\gamma_1 + \gamma_3\Omega^4} \tag{VI.10}$$

So that, by (VI.2), and using the fact that error in $\hat{x}(T)$ = error in $\hat{\mathcal{E}}_Q(T)$, we can say that, for the best estimate of position,

$$\text{(VI.11)} \qquad \lim_{T\to\infty} \overline{\left[\hat{x}(T) - x(T)\right]^2} = \frac{\gamma_1 \gamma_3}{\gamma_1 + \gamma_3 \Omega^4}$$

C. CASE WHERE $\tilde{\mathcal{E}}_V(t) = 0$

In this case

$$\text{(VI.12)} \qquad S''(t) + \Omega^2 S(t) = \Omega^2 \left(\mathcal{E}_{X_o} - \frac{K_B}{\Omega^2}\right) + K_V \Omega^2 t - \tilde{\mathcal{E}}_B(t)$$

Also

$$\text{(VI.13)} \qquad K_V = \lim_{T\to\infty} \frac{S(T)}{T}$$

and

$$\text{(VI.14)} \qquad \mathcal{E}_{X_o} - \frac{K_B}{\Omega^2} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left[\frac{S''(T)}{\Omega^2} + S(t) - K_V(t)\right] dt$$

The remainder of the discussion is the same as in the previous case. Eq. (VI.11) holds in this case also.

D. GENERAL CASE

In the general case it is not possible to separate $f_V(t) = \int_0^t \tilde{\mathcal{E}}_V(\tau)d\tau$ exactly from $f_B(t) = \int_0^t \tilde{\mathcal{E}}_B(\tau) \frac{\sin \Omega(t-\tau)}{\Omega} d\tau$ even as $t \to \infty$. Consequently the variance of $\hat{x}(T)$ as $T \to \infty$ will be larger than in the two special cases discussed above.

Let

$\Phi_V(\omega)$ = spectrum of $\tilde{\mathcal{E}}_V(t)$

$\Phi_B(\omega)$ = spectrum of $\tilde{\mathcal{E}}_B(t)$

Since $f_V(t)$ and $f_B(t)$ are not stationary, they do not possess spectra in the strict sense. However, heuristically one should think of $f_V(t)$, $f_B(t)$ as having spectra

$$\Psi_V(\omega) = \frac{\Phi_V(\omega)}{\omega^2} \quad ; \quad \Psi_B(\omega) = \frac{\Phi_B(\omega)}{(\omega^2 - \Omega^2)^2}$$

It seems reasonable to conclude that in the general case

$$\lim_{T\to\infty} \overline{\left[\hat{x}(T) - x(T)\right]^2} = \frac{\gamma_1 \gamma_3}{\gamma_1 + \gamma_3\Omega^4} + \text{Additional term} \qquad \text{(VI.15)}$$

where "Additional term" depends on $\Phi_V(\omega)$ and $\Phi_B(\omega)$ in such a way that it is small if the graphs of $\Psi_V(\omega)$ and $\Psi_B(\omega)$ do not overlap much:

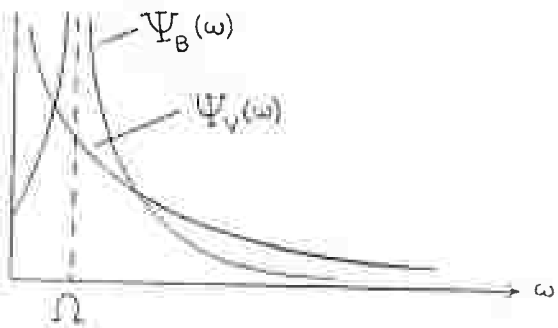

Fig. 2

Appendix I

EVALUATION OF LIMITS OF BILINEAR FORMS

The problem at hand is to evaluate $P = \lim_{n \to \infty} P_n$, where

$$P_n = \sum_{i,j=0}^{n-1} \xi_{ij} u_i y_j \tag{AI.1}$$

Here $u_i = u(t_i)$, $y_j = y(t_j)$; t_i, $i = 0, 1, \ldots, n-1$ are n equally-spaced points in the interval $(0,T)$ with $t_0 = 0$, $t_{n-1} = T$. $u(t)$ is a continuous function for $0 \leq t \leq T$, $y(t)$ is an entire function of t.

$$\text{Also} \quad \left[\xi_{ij}\right] = \left[\phi(t_i, t_j)\right]^{-1}$$

$$\text{where} \quad \phi(t_i, t_j) = \gamma + \beta e^{-\alpha|t_i - t_j|}$$

As pointed out in Ref. 4, pp. 295-296,

$$P = \int_{-0}^{T+0} u(t)\, w(t)\, dt \tag{AI.2}$$

where $w(t)$ is the solution of

$$y(t) = \int_{-0}^{T+0} \left[\gamma + \beta e^{-\alpha|s-t|}\right] w(s)\, ds \tag{AI.3}$$

To find the solution of (AI.3), rewrite (AI.3) as

$$y(t) - I = \int_{-0}^{T+0} \beta e^{-\alpha|s-t|}\, w(s)\, ds \tag{AI.4}$$

$$\text{where } I = \gamma \int_{-0}^{T+0} w(s)\, ds$$

If we forget for the moment that I actually depends on the unknown function w then (AI.4) is the same equation considered in Ref. 4. The solution is given in Ref. 4, Eq. 50:

$$\text{(AI.5)} \qquad w(t) = \frac{\alpha}{2\beta}\left[y(t) - I - \frac{1}{\alpha^2}y''(t)\right]$$

$$+ \frac{1}{2\beta}\left[y(0) - I - \frac{1}{\alpha}y'(0)\right]\delta\,(t-0)$$

$$+ \frac{1}{2\beta}\left[y(T) - I + \frac{1}{\alpha}y'(T)\right]\delta\,(t-T)$$

Now integrate both sides of (AI.5) over (-0,T+0) and solve for I. The result is

$$\text{(AI.6)} \qquad I = \frac{\alpha\int_0^T y(t)\,dt + y(0) + y(T)}{2 + \alpha T + \frac{2\beta}{\gamma}}$$

To obtain P, substitute (AI.6) into (AI.5), and then substitute the resulting expression for w(t) into (AI.2). The result is

$$\text{(AI.7)} \qquad P = \frac{\alpha}{2\beta}\int_0^T u(t)\left[y(t) - \frac{1}{\alpha^2}y''(t)\right]dt + \frac{1}{2\beta}u(0)\left[y(0) - \frac{1}{\alpha}y'(0)\right]$$

$$+ \frac{1}{2\beta}u(T)\left[y(T) - \frac{1}{\alpha}y'(T)\right]$$

$$- \frac{1}{2\beta}\,\frac{\left[\alpha T\,\tilde{u}(T) + u(0) + u(T)\right]\left[\alpha T\,\tilde{y}(T) + y(0) + y(T)\right]}{2 + \alpha T + \frac{2\beta}{\gamma}}$$

where $\tilde{u}(T) = \frac{1}{T}\int_0^T u(t)\,dt$, $\tilde{y}(T) = \frac{1}{T}\int_0^T y(t)\,dt$.

Appendix II

CASE 3: $\phi_B(s,t) = \beta_1 e^{-\alpha_1|s-t|}$; $\phi_V(s,t) = \beta_2 e^{-\alpha_2|s-t|}$

The method outlined in Section II for dealing with this case seems exceedingly difficult to carry through. Therefore, a method more amenable to analysis is used in the following; this method is still based on the maximum likelihood principle.

Consideration will be restricted to dial readings B(t), V(t) which are continuous functions. With the assumed autocorrelation functions, it is possible to do this.

The particular method used below of deriving the optimum position estimation process also requires that B(t) and V(t) have a certain number of derivatives. Now, with the assumed autocorrelation functions, B(t) and V(t) will be nondifferentiable with probability one. Hence the particular derivation given below will, strictly speaking, be valid for a set of functions B(t), V(t) having probability zero. However, the final answer--i.e. the method of obtaining $\hat{x}(T)$ from B(t), V(t)--involves operations on B(t) and V(t) which are valid for any continuous B(t), V(t). Also, any continuous function can be approximated in (0,T) as closely as desired by functions having any desired number of derivatives. These facts are assumed to imply that the final answer is the optimum--i.e. linear least squares--estimate of position for any continuous dial readings B(t), V(t). This amounts to assuming that if $B_k(t)$, $V_k(t)$ yield linear least squares estimates $\hat{x}_k(T)$; if B(t), V(t) yield a linear least squares estimate $\hat{x}(T)$; and if $B_k(t)$, $V_k(t)$ approach B(t), V(t) in (0,T) in some appropriate sense, then $\hat{x}(T)$ is the limit of $\hat{x}_k(T)$.

The gist of the above remarks is that various limiting processes will be interchanged when such seems intuitively justifiable.

As in Sections III and IV, the maximum likelihood method will be applied with Gaussian statistics. The result is then the linear least squares estimate for any statistics having the same means and correlation functions.

Consider the random vector $\vec{U} = (B_o, \ldots, B_{n-1}, V_o, \ldots, V_{n-1}, X_o, X_o')$, where $B_i = B(t_i)$; $V_i = V(t_i)$. The logarithm of the joint probability density of $\vec{U}$ is, apart from an additive constant, equal to $-Q_n\left[x(t)\right]$, where

$$\text{(AII.1)}\quad Q_n\left[x(t)\right] = \sum_{i,j=o}^{n-1} \xi_{ij}(V_i - x_i')(V_j - x_j') + \frac{1}{\gamma_3}(X_o - x_o)^2 + \sum_{i,j=o}^{n-1} \eta_{ij}(B_i - \mathcal{L}x_i)(B_j - \mathcal{L}x_j) + \frac{1}{\gamma_4}(X_o' - x_o')^2$$

Here $\left[\xi_{ij}\right] = \left[\phi_V(t_i, t_j)\right]^{-1}$

$$\left[\eta_{ij}\right] = \left[\phi_B(t_i, t_j)\right]^{-1}$$

The limit of this quantity as $n \to \infty$ can be evaluated by the method of Appendix I. The result is (after suitable integrations by parts)

$$\text{(AII.2)}\quad Q\left[x(t)\right] = \frac{\alpha_1}{2\beta_1}\int_o^T \left(u^2 + \frac{1}{\alpha_1^2}u'^2\right)dt + \frac{1}{2\beta_1}\left[u^2(0) + u^2(T)\right]$$
$$+ \frac{\alpha_2}{2\beta_2}\int_o^T \left(y^2 + \frac{1}{\alpha_2^2}y'^2\right)dt + \frac{1}{2\beta_2}\left[y^2(0) + y^2(T)\right]$$
$$+ \frac{1}{\gamma_3}(X_o - x_o)^2 + \frac{1}{\gamma_4}(X_o' - x_o')^2$$

where $u(t) = B(t) - \mathcal{L}x(t)$

$y(t) = V(t) - x'(t)$

To find the linear least squares estimate of $x(t)$, $0 \leq t \leq T$, based on the dial readings up to time T, one must minimize the quantity $Q\left[\hat{x}(t;T)\right]$ with respect to the function $\hat{x}(t;T)$.

This can be done by the usual Euler equation with free end conditions. The result is:

The Euler equation satisfied by $x(t;T)$ is, in symbolic notation,*

$$\text{(AII.3)}\qquad \left\{\frac{1}{\alpha_1\beta_1}\mathcal{L}^2\left(\mathcal{D}^2-\alpha_1^2\right)-\frac{1}{\alpha_2\beta_2}\mathcal{D}^2\left(\mathcal{D}^2-\alpha_2^2\right)\right\}\hat{x}(t;T)$$
$$=\left\{\frac{1}{\alpha_1\beta_1}\mathcal{L}\left(\mathcal{D}^2-\alpha_1^2\right)\right\}B(t)-\left\{\frac{1}{\alpha_2\beta_2}\mathcal{D}\left(\mathcal{D}^2-\alpha_2^2\right)\right\}V(t)$$

In addition, there are six end conditions involving the values of $\hat{x}(t;T)$, $B(t)$, $V(t)$, and their derivatives at $t = 0$ and $t = T$.

The simplest way to utilize the end conditions seems to be as follows:

The general solution to (AII.3) consists of certain integral transforms of B and V, plus an expression involving the initial values of $\hat{x}(t;T)$ and its first five derivatives, together with solutions of the homogeneous equation corresponding to (AII.3). The Euler end conditions can be used to boil this expression down to one involving only $\hat{x}(0;T)$, $\widehat{x'}(0;T)$, and $\widehat{\mathcal{L}x}(0;T) - B_0$. The function $\hat{x}(t;T)$ thus obtained can be re-substituted into (AII.2) and the resulting Q minimized with respect to $\hat{x}(0;T)$, $\widehat{x'}(0;T)$ and $\widehat{\mathcal{L}x}(0;T) - B_0$.

The final result can be expressed as follows:

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*All derivatives of $\hat{x}(t;T)$ referred to mean derivatives with respect to t, T being regarded as fixed.

$$\text{(AII.4)} \qquad \hat{x}(t;T) = \int_0^T \left[K_1(t-\tau)\, B(\tau) - K_2(t-\tau)\, V(\tau) \right] d\tau$$

$$+ \hat{x}(0;T)\, f_1(t) + \widehat{x'}(0;T)\, f_2(t)$$

$$+ \left[\widehat{\mathcal{L}x}(0;T) - B_o \right] f_3(t) + X_o\, f_4(t)$$

$$+ X_o'\, f_5(t) + V_o\, f_6(t)$$

where $f_1(t), \ldots, f_6(t)$ are solutions of the homogeneous equation corresponding to (AII.3). The kernels $K_1(t)$, $K_2(t)$ can best be given by giving their Laplace transforms:

(AII.5)

$$\text{Laplace transform of } K_1(t) = \frac{\frac{1}{\alpha_1\beta_1}\left(p^2+\Omega^2\right)\left(p^2-\alpha_1^2\right)}{\frac{1}{\alpha_1\beta_1}\left(p^2+\Omega^2\right)^2\left(p^2-\alpha_1^2\right) - \frac{1}{\alpha_2\beta_2}\, p^2\left(p^2-\alpha_2^2\right)}$$

$$\text{Laplace transform of } K_2(t) = \frac{\frac{1}{\alpha_2\beta_2}\, p\left(p^2-\alpha_2^2\right)}{\frac{1}{\alpha_1\beta_1}\left(p^2+\Omega^2\right)^2\left(p^2-\alpha_1^2\right) - \frac{1}{\alpha_2\beta_2}\, p^2\left(p^2-\alpha_2^2\right)}$$

The quantities $\hat{x}(0;T)$, $\widehat{x'}(0;T)$ and $\widehat{\mathcal{L}x}(0;T) - B_o$ are the solution to a set of linear simultaneous equations:

$$\text{(AII.6)} \qquad \sum_{k=1}^{3} A_{jk}\, z_k = D_j \qquad j = 1,2,3$$

where $\quad z_1 = \hat{x}(0;T)\ ;\ z_2 = \hat{x}'(0;T)\ ;\ z_3 = \widehat{\rho x}(0;T) - B_o$

The quantities A_{jk} depend on T and on the functions $f_1, \dots, f_6$ (as well as α_1, α_2, β_1, β_2); D_j depend on these and on the dial readings X_o, Y_o', $B(t)$, $V(t)$, $0 \leqq t \leqq T$.

All operations involved in finding $\hat{x}(t;T)$ are valid for any continuous functions $B(t)$ and $V(t)$.

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